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Evaluation of Drag Integral Using Cubic Splines

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Introduction

THE well-known wave drag integral[1] which expresses the drag of suitably slender configurations in terms of an area distribution $S(x)$ can be written

$$I = -\frac{1}{2\pi}\int_0^1\int_0^1 S''(x)S''(y)\ \ln\ |x-y|\,dxdy \tag{1}$$

This integral has been studied extensively in the past, notably by Eminton[2] to whom goes credit for the most widely used practical method of evaluation, given only a discrete point data set of values of $S(x_i)$. For most engineering applications, the input data set is quite crude whether the values represent areas or some other physical quantity arising from different calculations to which I applies. Nevertheless, $S(x)$ must be assumed to satisfy certain conditions, and the minimum necessary for Eq. (1) to be useful are:

$$\begin{aligned} &S'(x) \quad \text{is continuous in} \quad (0 \le x \le 1) \\ &S'(0) = S'(1) = 0 \end{aligned} \tag{2}$$

For the evaluation of drag when these conditions are not satisfied (in particular when $S'(x)$ is not continuous) more involved formulas than Eq. (1) alone are required,[3] but the restrictions, Eq. (2), are commonly assumed in practice and are required by Eminton's method.

Spline Techniques

With regard to the use of cubic (or other) splines in practical engineering, it is the authors' experience that all too often the initial enthusiasm is rapidly damped by the necessity for providing end conditions beyond mere point values in order to solve the usual tridiagonal matrix for the point second derivatives. If these extra conditions are not known with the same degree of precision as the point values of the function itself, then the spline formulation is no better than other simpler means of ad hoc interpolation. Many engineering applications start with poor quality input information, and under these circumstances splines are just another tool, not a panacea.

However, the conditions (2) are ready made for a spline assumption since the end derivatives are given exactly. Therefore, it was decided to test the accuracy of a spline method of evaluation of I by fitting the area distribution with a cubic spline using a standard subroutine for rapid solution of the usual tridiagonal matrix—essentially by Gaussian elimination. Such a method, it was felt, might be quicker than Eminton's and would not require the inversion of a full matrix and so might not be subject to the usual problems arising from large full matrix operations (e.g., large computer core requirements).

This idea, obvious as it is, had not attracted much attention until recently. Halfway through the runs described in this Note, the work of V. V. Shanbhag[4] came to the authors' attention; but it was decided to publish these results anyway as a further source of comparison cases.

It should be noted that the Eminton method has a design aspect in the sense that the eventual interpolation provides a body of minimum drag subject to the given constraint of having specified areas at certain x stations. Such a facility cannot necessarily be claimed for the spline method, but the purpose of this Note is merely to compare some evaluations. A more comprehensive view of these methods must be left to the practicing designers in this field.

Calculation Procedure

If $S(x)$ is given as a polynomial in x subject to conditions (2), then it is easy to evaluate I analytically in terms of the numbers

$$I_{r,s} = -\frac{1}{2\pi}\int_0^1\int_0^1 x^r y^s\ \ln\ |x-y|\,dxdy$$

These have been computed at Douglas Aircraft Co. using the IBM 370/165 system and stored and tabulated (for quick hand estimates). By using these numbers an exact solution is readily available to provide a standard for comparison between the Eminton and spline methods. The Eminton calculations were obtained from a standard production subroutine using point values of $S(x)$ as input. For the spline calculations $S''(x)$ is piecewise linear and so with some tedious but minor manipulations I can be expressed in various condensed forms. It appears that the neatest involves fourth differences of $S_j = S(x_j)$, but from a computational accuracy point of view, the formula actually used had more of an "influence function" character, viz.:

$$I = \frac{3a_m a_1}{4\pi} - \frac{a_m}{\pi}\sum_{j=1}^{m-1} b_j[W_{j,m}] - \frac{a_1}{\pi}\sum_{j=1}^{m-1} b_j[W_{j,1}] - \frac{1}{2\pi}\sum_{k=1}^{m-1}\sum_{j=1}^{m-1} b_k b_j\langle Z_{j,k}\rangle$$

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Table 1 Coefficients for various cases (negative signs above numbers)

Case	p	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
1	2	0.5	$0.\overline{3}33\ldots$									
2	2	108	$5\overline{8}8$	1257	$1\overline{1}76$	400						
3	2	22.183	$1\overline{43}.098$	332.734	$3\overline{19}.686$	108.737						
4	2	13	$9\overline{7}$	295	$3\overline{1}5$	110						
5	6	6,400	$2\overline{5},600$	16,000	67,200	$12\overline{4},400$	34,400	101,200	$12\overline{4},800$	6,400	$1\overline{6},000$	1,600

where the a_j are values of the second derivative of S at the m stations $j = 1, \ldots m$ transferred directly from the spline subroutine, and $b_j = (a_{j+1} - a_j)/(x_{j+1} - x_j)$. The functions W and Z are indefinite integrals of $\ln |u|$ where $u = x - y$, in fact,

$$W(u) = (u^3/6) \ln |u| - (11u^3/36)$$

$$Z(u) = (u^4/24) \ln |u| - (25u^4/288)$$

and the brackets imply corner differences

$$[W_{j,m}] = W(x_{j+1} - 1) - W(x_j - 1), \quad (x_m = 1)$$

$$\langle Z_{j,k} \rangle = Z_{j+1,k+1} - Z_{j+1,k} - Z_{j,k+1} + Z_{j,k}$$

These influence numbers could be precomputed and stored except for the fact that none of these methods is necessarily restricted to equal intervals in the x_j, so that there is very little gain for arbitrary runs.

Description of Some Test Cases

Results are presented in tabular form for a certain sequence of polynomials characterized by

$$S(x) = x^p\{A_0 + A_1x^2 + \ldots A_qx^q\}, \quad p \geq 2$$

and listed according to the coefficients in Table 1. Of course Case 1 is trivial and Case 2 was originally offered by Eminton.[2] The other cases represent attempts to obtain more wavy or peaky distributions in order to provide more severe test runs. Cases 3 and 4 are appreciably different from Eminton's distribution while Case 5 has two peaks. As an illustration, Fig. 1 shows Case 5 together with a reasonable and unreasonable uneven point distribution. It is not particularly easy to generate sensible looking cases with $S(x) > 0$ everywhere by juggling with $S'(x)$ without undertaking a much more detailed investigation.

Comparison of Eminton and Spline Results

Comparisons are given in terms of percentage error:

$$\text{Error} = \frac{100 \times (\text{computed value} - \text{exact})}{\text{exact}}$$

for equal interval inputs, reasonably chosen varying input stations, and unreasonably chosen varying input stations. It can be seen from Fig. 1 that the kind of difference implied by these choices for Case 5 is extreme. For the purposes of testing, the programs used double precision arithmetic and points were generated internally where necessary for accuracy. For all the cases quoted here, computing times were small enough and similar enough to make precise comparisons by disentangling various aspects of computer charges not worth the effort. As can be seen from Table 2, the spline estimates were very much better than the Eminton, in the majority of cases the errors being from one tenth to one hundredth for somewhat less expenditure of computer time.

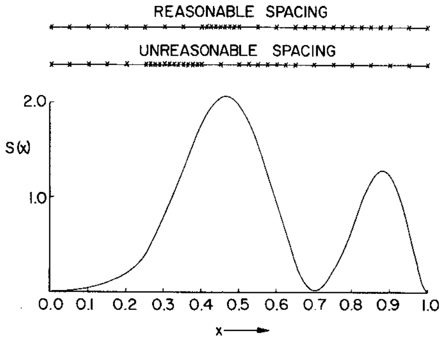

Fig. 1 Case 5 showing choices of uneven spacing used.

Table 2 Equal interval input

Case	Number of points	Error: Eminton	Error: Spline
1	2	−11.1	0
1	21	−0.085	0
2	21	−1.594	+0.069
2	33	−0.637	+0.013
2	51	−0.256	+0.003
3	21	−0.810	+0.030
3	51	−0.128	+0.000
4	11	−0.337	−0.078
4	21	−0.048	−0.004
4	51	−0.005	−0.000
5	21	−2.231	+0.153
5	51	−0.405	+0.007

Table 3 Reasonably chosen unequal intervals

Case	Number of points	Error: Eminton	Error: Spline
5	32	−2.000	+0.195
5	51	−0.304	+0.008

Table 4 Unreasonably chosen unequal intervals

Case	Number of points	Error: Eminton	Error: Spline
5	33	−2.203	+0.159
5	51	−0.460	−0.035

For unequal but reasonably chosen intervals such that more data points were concentrated [by variation with $S''(x)$] in regions of highest curvature results were as shown in Table 3. As can be seen, these differ little from the equal interval cases shown in Table 1 and the spline maintains its greater accuracy.

Finally, for unreasonably chosen unequal spacing, such that points were concentrated where not required and thinned out in high variation regions, the corresponding results were as shown in Table 4. Evidently the unreasonable choice exercised for Table 4 was not too severe; it was supposed to represent a case where unequal intervals were forced on the designer and not to his liking, so it should not be ridiculous. Neither method suffers excessively through this imposition; however it is clear that the spline is more sensitive to it. The apparent improvement in the case of the spline is presumably fortuitous for the 33-point choices since in the 51 point cases the drag error changes sign.

Conclusions

On the basis of these somewhat sketchy test cases, both methods show the expected kind of convergence as the number of points increases. Furthermore, the two methods tend to produce errors of opposite sign since Eminton underestimates the drag (as would be expected from the nature of the method) whereas it appears that the spline assumption has a tendency to overestimate.

It is not suggested that the spline method should replace Eminton, which has various aspects important in practical work—such as the design feature. However, troubles have been known in using this method and for this reason, it is suggested that the spline would be a valuable adjunct in design evaluation work. It has the potential for more accurate and quicker computation of the drag integral as indicated by the preceding studies. Nevertheless, these studies were based on polynomials only, from which one can fairly infer that they were biased in favor of the spline method. Clearly more work needs to be done, but the above results suggest that it is certainly worth doing.

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